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PERIODIC SOLUTIONS TO A LIMITED PROBLEM OF THREE BODIES, INCLUDING A LARGE NUMBER OF REVOLUTIONS AROUND A SMALL BODY

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16. Abstract

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PERIODIC SOLUTIONS TO A LIMITED PROBLEM OF THREE BODIES, INCLUDING A LARGE NUMBER OF REVOLUTIONS AROUND A SMALL BODY

I. V. Kurcheyeva

To construct periodic solutions in a limited problem of three bodies, the method of the small Poincare parameter is widely used. In a system of differential equations, terms are dropped which contain the small parameter. Given a new system which is called generative, that admits periodic solutions. In executing several conditions, one and only one periodic solution of the primary system will satisfy the periodic solution of the generative system. In striving to zero of the small parameter, the periodic solution approaches the generative. /168

There are several methods for constructing different generative systems in a limited problem of three bodies.

1. Terms are discarded which depend on the perturbing mass; consequently we derive a problem of two bodies (this method is for deriving periodic solution of a limited problem of three bodies and was first employed by Poincare [1], then Schwarzshild [2], Batrakov [3-4], Arenstorf [5-8] and many others).

2. The mean motion n is taken as the small parameter. The generative system will be a problem of two stationary centers (cf. Demin [9], Arenstorf [6]).

3. The small parameter is introduced artificially by transformation of coordinates and time (cf. Aksënov [10]).

4. The averaged problem of three bodies is taken as the

generative system (the method introduced by Merman for the averaged Fatu problem [12]).

In this article, the averaged Gaussian problem is taken as the generative system.

As we know, a plane limited circular problem of three bodies/169 studies the motion of point P of infinitely small mass under the effect of Newtonian gravitation of two material points E and M, rotating around their common center of gravity in circular orbits. The units of time, mass and distance are selected so that the mass of bodies E and M would be equated respectively to $1 - \mu$, μ and their reciprocal distances, angular velocity, gravitational constant would be equated to one.

Then, in a polar system of coordinates (body M of lesser mass is taken as the center) the equations of motion of point P will be written in the form

$$\left.\begin{aligned} \ddot{r} &= r\dot{\varphi}^2 + W'_r, \\ \frac{d}{dt}(r^2\dot{\varphi}) &= W'_\varphi, \end{aligned}\right\} \qquad (1)$$

where W is the force function describable by formula

$$W = \frac{\mu}{r} + (1-\mu)\left(\frac{1}{\sqrt{1+r^2-2r\cos\theta}} - r\cos\theta\right), \qquad (2)$$

where θ is the difference in longitudes of ϕ_E, ϕ of point E and P respectively, which is equal to

$$\theta = \varphi - \varphi_E.$$

Since the average motion n_E of point E is equal to one, then $\phi_E = t$, and

$$\theta = \varphi - t. \qquad (3)$$

System (1) has a unique integral (Jacobi integral) which we can write in final form

$$\dot{r}^2 + r^2(\dot{\varphi} - 1)^2 = 2\left[W + \frac{1}{2}r^2\right] + 2h. \tag{4}$$

In Keplerian phase space (a, p, M,ω), equations (1) have the form

$$\begin{aligned} \frac{da}{dt} &= \frac{2\sqrt{a}}{\sqrt{\mu}} \cdot \frac{\partial R}{\partial M}, \\ \frac{dp}{dt} &= \frac{2\sqrt{p}}{\sqrt{\mu}} \cdot \frac{\partial R}{\partial \omega}, \\ \frac{dM}{dt} &= \frac{\sqrt{\mu}}{a^{3/2}} - \frac{2\sqrt{a}}{\sqrt{\mu}} \cdot \frac{\partial R}{\partial a}, \\ \frac{d\omega}{dt} &= -\frac{2\sqrt{p}}{\sqrt{\mu}} \cdot \frac{\partial R}{\partial p}. \end{aligned} \tag{5}$$

Here R is the perturbation function which, in a polar system of coordinates, has the form

$$R = (1-\mu)\left[\frac{1}{\sqrt{1+r^2-2r\cos\theta}} - r\cos\theta\right]. \tag{6}$$

By the Keplerian phase coordinates R we can express, with the aid of formulas

$$\begin{gathered} r = a\left[1 - \sqrt{1 - \frac{p}{a}}\right]\cos E, \\ \varphi = \omega + v, \\ \operatorname{tg}\frac{v}{2} = \sqrt{\frac{a}{p}}\left[1 + \sqrt{1 - \frac{p}{a}}\right]\operatorname{tg}\frac{E}{2}, \\ E - \sqrt{1 - \frac{p}{a}}\sin E = M. \end{gathered} \tag{7}$$

To eliminate the possible collision of planetoid P and body E, let us introduce the condition /170

$$a\left(1+\sqrt{1-\frac{p}{a}}\right)<1. \tag{8}$$

Inequality (8) denotes that body P, in all its time of motion, does not go beyond the orbit of body E.

Then function R can be expanded into a Fourier series:

$$R=\sum_{q=0}^{\infty}\sum_{r=-\infty}^{\infty} C_{q,r}(a, p)\cos(qM+r\bar{\omega}), \tag{9}$$

where $\bar{\omega} = \omega - t$, and $C_{q,r}$ are defined functions of a, p. Let us assume that

$$R=R_1(a, p)+R_2(a, p, M, \bar{\omega}). \tag{10}$$

This presentation is always possible, since R is a periodic function of M and t with a period of 2π, 2π and for R_1 we can take a twice averaged function of R:

$$R_1=C_{00}(a, p)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} R\,dM\,dt. \tag{11}$$

In this case, function R_2 will possess the following property: in terms of any a priori given small $\varepsilon > 0$ we will always find such a small δ-neighborhood of body M ($r < \delta$) in which the inequality $|R_2| < \varepsilon$ will be fulfilled. This follows from the continuity of function R_2 and the fact that $R_2(0, 0, M, \bar{\omega}) = 0$.

In the small δ-neighborhood of body M, the function can be written as

$$R_2=\sigma\tilde{R}_2, \tag{12}$$

where σ is a small parameter, and $R_2 = \frac{\tilde{R}_2}{l}$, $l = \sigma$.

We will seek a solution of system (5) which lies totally in the small δ-neighborhood of point M. In this case, we can apply to system (5) the method of the Poincare small parameter.

Where $\delta = 0$, $i \neq 0$, system (5) will appear as:

$$\begin{aligned} &\frac{da}{dt} = 0, \\ &\frac{dp}{dt} = 0, \\ &\frac{dM}{dt} = \frac{\sqrt{\mu}}{a^{3/2}} - \frac{2\sqrt{a}}{\sqrt{\mu}} \cdot \frac{\partial R_1}{\partial a}, \\ &\frac{\partial \omega}{\partial t} = -\frac{2\sqrt{p}}{\sqrt{\mu}} \cdot \frac{\partial R_1}{\partial p}. \end{aligned} \tag{13}$$

We have a twiced averaged Gaussian problem, integrating which yields

$$\begin{aligned} &a = a_0, \\ &p = p_0, \\ &M = n_1 t + M_0, \\ &\omega = n_2 t + \omega_0, \end{aligned} \tag{14}$$

where /171

$$\begin{aligned} &n_1 = \frac{\sqrt{\mu}}{a^{3/2}} - \frac{2\sqrt{a}}{\sqrt{\mu}} \cdot \frac{\partial R_1}{\partial a}, \\ &n_2 = -\frac{\sqrt{p}}{\sqrt{\mu}} \cdot \frac{\partial R_1}{\partial p}. \end{aligned} \tag{15}$$

Equations (14) are the motion along the ellipses with focus at point M. The semiaxis and parameter are invariable, and the apsidal lines rotate with constant angular velocity, dependent only on a and p. The law of dependence of M on time remains as in unperturbed motion linear, but with a variable coefficient of proportionality. The trjaectory of motion relative the stationary axes will be periplegmatic trajectory, spinning between two circles, first touching one, then the other. The centers of the circles are located at point M, and the radii are equal, respectively, to

$$r_\pi = a\left(1 - \sqrt{1 - \frac{p}{a}}\right),$$
$$r_\alpha = a\left(1 + \sqrt{1 - \frac{p}{a}}\right).$$

If constants $a_0 p_0$ satisfy the condition

$$\begin{gathered} n_1 T = 2k_1\pi, \\ (n_2 - 1)\, T = 2k_2\pi, \end{gathered} \tag{16}$$

where k_1, k_2 are whole integers, then solution of (14) can be considered periodic, of period T.

Let us investigate only such constants a_0, p_0 for which are fulfilled the inequality

$$a_0\left(1 + \sqrt{1 - \frac{p_0}{a_0}}\right) < \delta. \tag{17}$$

Under such conditions, periodic solutions of the system of differential equations (13) will lie wholly wtihin the δ-neighborhood of point M.

According to Poincare's method, the periodic solution of system (5), close to (14), and having the same period, will be written as

$$\begin{aligned} a &= a_0+\beta_1+\xi_1, \\ p &= p_0+\beta_2+\xi_2, \\ M &= n_1 t+\gamma_1+\eta_1, \\ \omega &= n_2 t+\gamma_2+\eta_2, \end{aligned} \tag{18}$$

ξ_i, η_i are new unknown periodic functions of period T:

$$\xi_i(0)=0, \qquad \eta_i(0)=0. \tag{19}$$

To define the small quantities ξ_i, γ_i, let us write the equations

$$\begin{aligned} &M(0)=0,\ \bar{\omega}(0)=0, \\ &M\left(\frac{T}{2},\ \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma\right)=\pi k_1, \\ &\bar{\omega}\left(\frac{T}{2},\ \beta_1, \beta_2, \gamma_1, \gamma_2, \sigma\right)=\pi k_2. \end{aligned} \tag{20}$$

These conditions signify that the curve described by equations (18) becomes symmetrical relative to the axis connecting both bodies E and M and is closed. /172

The conditions of periodicity of (20) are taken instead of the classic condition of rotation of a point in an initial state through a time interval T > 0, which leads to the expressed case.

Substituting (18) in (5), let us define η_1 and η_2 to within terms of the first power σ, β_i, γ_i:

$$\begin{aligned} \eta_1\left(\frac{T}{2}\right) &= \left(\frac{\partial n_1}{\partial a_0}\beta_1+\frac{\partial n_1}{\partial p_0}\beta_2\right)\frac{T}{2}+\dots, \\ \eta_2\left(\frac{T}{2}\right) &= \left(\frac{\partial n_2}{\partial a_0}\beta_1+\frac{\partial n_2}{\partial p_0}\beta_2\right)\frac{T}{2}+\dots \end{aligned} \tag{21}$$

Let us rewrite (20), using (18) and (21):

$$\begin{gathered}\gamma_1=0, \quad \gamma_2=0, \\ \left(n_1+\frac{\partial n_1}{\partial a_0}\beta_1+\frac{\partial n_1}{\partial p_0}\beta_2\right)\frac{T}{2}+\ldots=2\pi k_1, \\ \left(n_2+\frac{\partial n_2}{\partial a_0}B_1+\frac{\partial n_2}{\partial p_0}\beta_2-1\right)\frac{T}{2}+\ldots=2\pi k_2.\end{gathered} \tag{22}$$

n_1, n_2 are linearly independent functions, because

$$D=\begin{vmatrix}\frac{\partial n_1}{\partial a_0} & \frac{\partial n_1}{\partial p_0} \\ \frac{\partial n_2}{\partial a_0} & \frac{\partial n_2}{\partial p_0}\end{vmatrix}\neq 0.$$

And consequently, there is always an infinite set a_0, p_0 whose determinant D is not equal to zero. Equations (22) are solvable solely with respect to β_1, β_2 in the form of holomorphous functions with respect to σ.

Consequently, for small $\sigma > 0$, there exist periodic solutions which are symmetrical relative to the axis connecting bodies E and M which, when $\sigma = 0$, are in conformity with periodic solutions of (14), (16) of the generative system.

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